Chaos in FRW cosmology with gently sloping scalar field potentials

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Abstract

The chaotic behavior in FRW cosmology with a scalar field is studied for scalar field potentials less steep than quadratic. We describe a transition to much stronger chaos for appropriate parameters of such potentials. The range of parameters which allows this transition is specified. The influence of ordinary matter on the chaotic properties of this model is also discussed.

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1 Introduction

In the last few years the chaotic regime in dynamics of closed FRW universe filled with a scalar field becomes the issue of investigations. Initially, the model with a massive scalar field (with the scalar field potential $V(\varphi) = (m^2\varphi^2)/2$, where m is the mass of the scalar field) was studied [1, 2]. Before summarizing main results obtained, we present the equation of motion (for further using we will not specify the potential $V(\varphi)$). The system has two dynamical variables - the scale factor a and the scalar field φ:

$$\frac{m_P^2}{16\pi}\left(\ddot{a} + \frac{\dot{a}^2}{2a} + \frac{1}{2a}\right) + \frac{a\dot{\varphi}^2}{8} - \frac{aV(\varphi)}{4} = 0, \tag{1}$$

$$\ddot{\varphi} + \frac{3\dot{\varphi}\dot{a}}{a} + V'(\varphi) = 0. \tag{2}$$

with the first integral

$$-\frac{3}{8\pi}m_P^2(\dot{a}^2 + 1) + \frac{a^2}{2}\left(\dot{\varphi}^2 + 2V(\varphi)\right) = 0. \tag{3}$$

Here m_P is the Planck mass.

The points of maximal expansion and those of minimal contraction, i.e. the points, where $\dot{a} = 0$ can exist only in the region where

$$a^2 \leq \frac{3}{8\pi}\frac{m_P^2}{V(\varphi)}, \tag{4}$$

Sometimes, the region defined by inequalities (4) is called the Euclidean one. One can easily see also that the possible points of maximal expansion (where $\dot{a} = 0$, $\ddot{a} < 0$) are localized inside the region

$$a^2 \leq \frac{1}{4\pi}\frac{m_P^2}{V(\varphi)} \tag{5}$$

while the possible points of minimal contraction (where $\dot{a} = 0$, $\ddot{a} > 0$) lie outside this region (5) being at the same time inside the Euclidean region (4).

The main idea of the further analysis [3] consists in the fact that in the closed isotropical model with a minimally coupled scalar field satisfying the energodominance condition all the trajectories have the point of maximal expansion. Then the trajectories can be classified according to localization of their points of maximal expansion. The area of such points is specified by (5). A numerical investigation shows that this area has a quasi- periodical structure, wide zones corresponding to the falling to singularity being intermingled with narrow those in which the points of maximal expansion of trajectories having the so called "bounce" or point of minimal contraction are placed. Then studying the substructure of these zones from the point of view of possibility to have two bounces one can see that this substructure reproduce on the qualitative level

the structure of the whole region of possible points of maximal expansion. Continuing this procedure *ad infinitum* yields the fractal set of infinitely bouncing trajectories.

It should be noticed that even the 1-st order bounce intervals (containing maximum expansion points for trajectories having at least one bounce) are very narrow. Analytical approximation for large initial a indicates that the width of intervals is roughly inversely proportional to a [4]. The opposite case of small initial a was investigated numerically, and the ratio of the first such interval width to the distance between intervals appear to be of the order of 10^{-2}, if we do not take into account zigzag-like trajectories. So, the chaotic regime, though being interesting from the mathematical point of view, may be treated as not important enough.

For steeper potentials the chaos is even less significant. The chaotic behavior may disappear completely for exponentially steep potentials [5] .

The goal of the present paper is to describe the opposite case - the potentials which is less steep than the quadratic one. We will see that in this case the transition to a qualitatively stronger chaos may occur.

The structure of the paper is as follows. It Sec.2 we consider asymptotically flat potential and explain new features of the chaos which give rise in this case. In Sec.3 a more wide class of potentials less steep than quadratic is studied. In Sec.4 we discuss the transition to regular dynamics in the presence of ordinary matter in addition to the scalar field for potentials under consideration.

2 Asymptotically flat potentials and the merging of bounce intervals

We will use the units in which $m_P/\sqrt{16\pi} = 1$ for presenting our numerical results, because in these units most of the interesting events occur for the range of parameters of the order of unity.

We start with the potential

$$V(\varphi) = M_0^4(1 - exp(-\frac{\varphi^2}{\varphi_0^2})), \tag{6}$$

where M_0 and φ_0 are parameters. M_0 determines the asymptotical value of the potential for $\varphi \to \pm\infty$.

It can be easily checked from the equations of motion that multiplying the potential to a constant (i.e. changing the M_0) leads only to rescaling a. So, this procedure do not change the chaotic properties of our dynamical system. On the contrary, this system appear to be very sensitive to the value of φ_0. We plotted in Fig.1. the $\varphi = 0$ cross-section of bounce intervals depending on φ_0. This plot represents a situation, qualitatively different from studied previously for potentials like $V \sim \varphi^2$ and steeper. Namely, the bounce intervals can merge.

Let us see more precisely what does it means. For $\varphi_0 > 0.82$ the picture is qualitatively as for a massive scalar field - trajectories from 1-st interval have a bounce with no φ-turns before it, trajectories which have initial point of maximal expansion between 1-st and 2-nd intervals fall into a singularity after one φ-turn, those from 2-nd interval have a bounce after 1 φ-turn and so on. For φ_0 a bit smaller than the first merging value the 2-nd interval contains trajectories with 2 φ-turns before bounce, the space between 1-st interval (which is now the product of two merged intervals) and the 2-nd one contains trajectories falling into a singularity after two φ-turns. There are no trajectories going to a singularity with exactly one φ-turn. Trajectories from the 1-st interval can
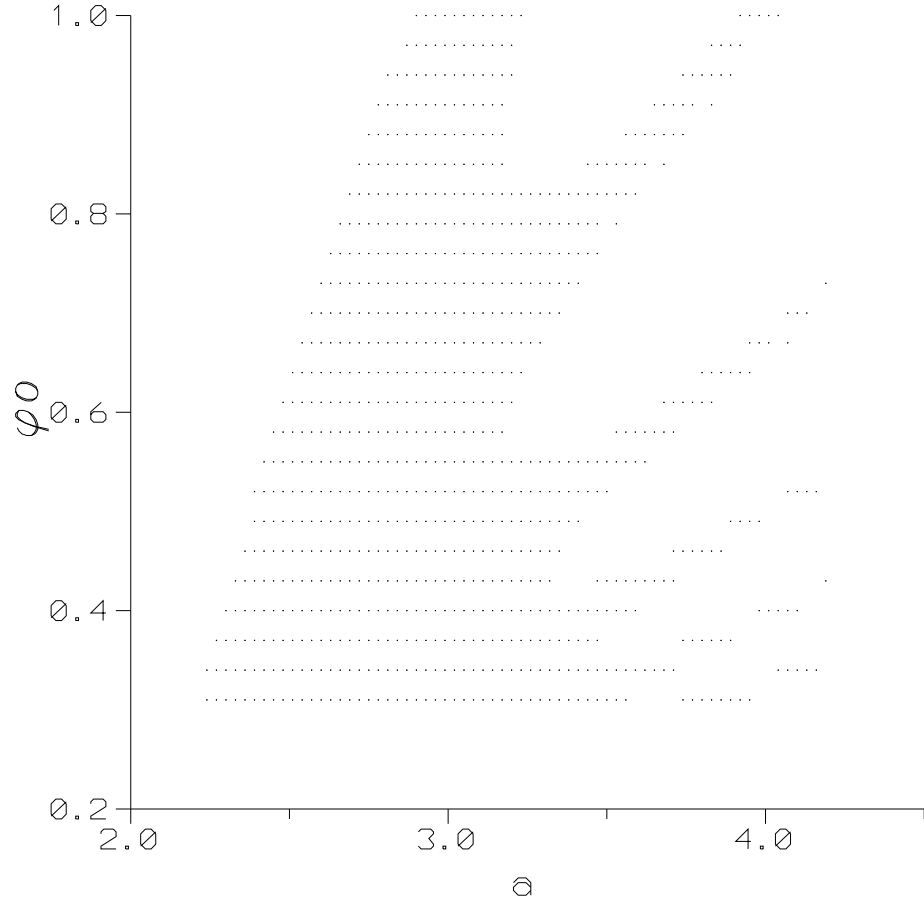
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Figure 1: The $\varphi = 0$ cross-section of the bounce intervals for the potential (6) depending on φ_0. Consecutive merging of 5 first intervals can be seen in this range of φ_0.

experience now a complicated chaotic behavior which can not be described in as similar way as above.

With φ_0 decreasing further, the process of interval merging being to continue leading to growing chaotisation of trajectories. When n intervals merged together, only trajectories with at least n oscillations of the scalar field before falling into a singularity are possible. Those having exactly n φ-turns have their initial point of maximal expansion between 1-st bounce interval and the 2-nd one (it now contains trajectories having a bounce after n φ-turns). For initial values of the scale factor larger then those from the 2-nd interval, the regular quasiperiodic structure described above is restored.

Numerical analysis shows also that the fraction of very chaotic trajectories as a function of φ_0 grows rapidly with φ_0 decreasing below the first merging value. To illustrate this point we plotted in Fig.2 the number of trajectories which do not fall into a singularity during first 50 oscillations of the scalar field φ. We do not include trajectories with the next point of maximal expansion located outside the 2-nd (or the 1-st one, if merging occurred) interval, so all counted trajectories avoid a singularity during this sufficiently long time interval due to their extreme chaoticity, but not due to reaching the slow-roll regime. The initial value of a vary in the range of the first two intervals before and after merging with the step 0.002. Before merging, the measure of so chaotic trajectories is extremely low and they are undistinguisheble on our grid. When φ_0 becomes slightly low than the value of the first merging, this number begin to grow rather rapidly and for $\varphi_0 \sim 0.6$ near 10% of trajectories from the 1-st interval experience at least 50 oscillation before falling into a singularity.

We recall that for a simple massive scalar field potential only $\sim 10^{-2}$ trajectories in the same range of the initial scale factors have at least one bounce.
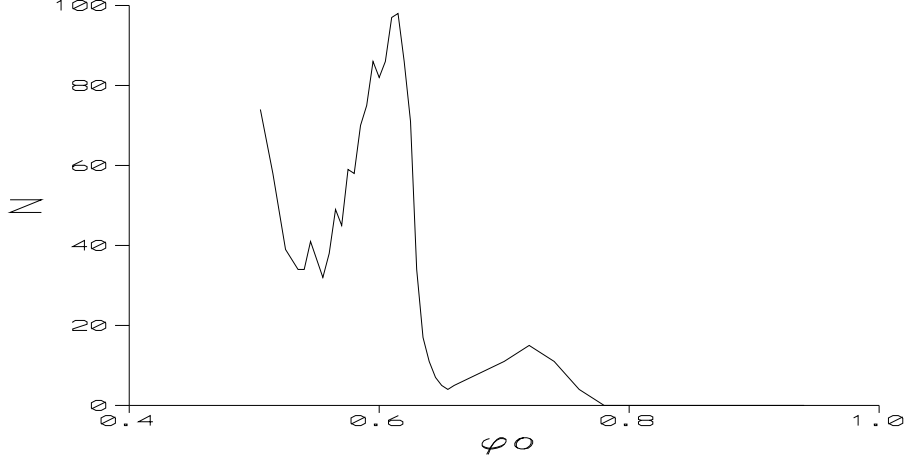
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Figure 2: Number N of trajectories do not falling into a singularity during 50 oscillating times for the potential (6) depending on the parameter φ_0. The scale factor of the initial maximal expansion point varies in the range of the 1-st and 2-nd intervals which merge at $\varphi_0 = 0.82$. Total number of trajectories is equal to 1000.

Fraction of trajectories not falling into a singularity after only one bounce is about one hundred times less and so on. The common numerical calculation accuracy is unsufficient for distinguishing even the sole trajectory with 50 oscillations and a being in the range of first two intervals.

In contrast to this, the chaos for the potential (6) is really significant. Detail of intervals merging including the description out of $\varphi = 0$ cross-section require further analysis.

For large initial a the configuration of bounce intervals for potential (6) looks like the configuration for a massive scalar field potential with the effective mass easily derived from (6): $m_{eff} = (\sqrt{2}M_0^2)/\varphi_0$. The periods of corresponding structures coincides with a good accuracy though the widths of the intervals for the potential (6) is bigger then for $V = (m_{eff}^2\varphi^2)/2$.

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3   Damour-Mukhanov potentials

The very chaotic regime described above is possible also for potentials, which are not asymptotically flat, if the potential growth is slow enough. We will illustrate this point describing a particular (but rather wide) family of potentials having power-low behavior – Damour - Mukhanov potentials [6]. They was originally introduced to show a possibility to have an inflation behaviour without slow-roll regime. After, various issues on inflationary dynamics [7] and growth of perturbation [8, 9] for this kind of scalar field potential was studied.

The explicit form of Damour-Mukhanov potential is

$$V(\varphi) = \frac{M_0^4}{q}\left[\left(1 + \frac{\varphi^2}{\varphi_0^2}\right)^{q/2} - 1\right],\tag{7}$$

with three parameters $-M_0$, q and φ_0.

For $\varphi \ll \varphi_0$ the potential looks like the massive one with the effective mass $m_{eff} = M_0^2/\varphi_0$. In the opposite case of large φ it grows like φ^q.

As in the previous section, the chaotic behavior does not depend on M_0. So, we have a two-parameter family of potentials with different chaotic properties. Numerical studies with respect to possibility of bounce intervals merging shows the following picture (see Fig.3): for a rather wide range of q there exists a corresponding critical value of φ_0 such that for φ_0 less than critical, the very chaotic regime exists. Increasing q corresponds to decreasing critical φ_0.

Surely, since this regime is absent for quadratic and more steep potentials, q must at least be less than 2. We can see clearly the very chaotic regime for $q < 1.24$. The case $q = 1.24$ lead to strong chaos for $\varphi_0 < 1.4 \times 10^{-5}$ and the critical φ_0 decreases with increasing q very sharply at this point. We did not investigated further these extremely small values of φ_0, because the physical meaning of such kind of potential is very doubtful.
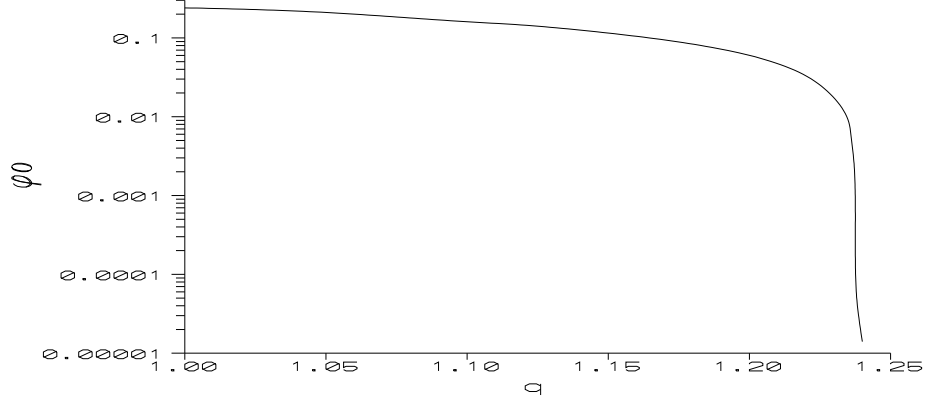
Figure 3: The value φ_0 of the potential (7) corresponding to the first merging of the bounce intervals depending on q.

4 The influence of a hydrodynamical matter

In this section we add the perfect fluid with the equation of state $P = \gamma\epsilon$. The equation of motion are now

$$\frac{m_P^2}{16\pi}\left(\ddot{a} + \frac{\dot{a}^2}{2a} + \frac{1}{2a}\right) + \frac{a\dot{\varphi}^2}{8} - \frac{V(\varphi)}{4} - \frac{Q}{12a^{p+1}}(1-p) = 0 \qquad (8)$$

$$\ddot{\varphi} + \frac{3\dot{\varphi}\dot{a}}{a} + V'(\varphi) = 0. \qquad (9)$$

with the constraint

$$-\frac{3}{8\pi}m_P^2(\dot{a}^2 + 1) + \frac{a^2}{2}\left(\dot{\varphi}^2 + 2V(\varphi)\right) + \frac{Q}{a^p} = 0. \qquad (10)$$

Here $p = 1 + 3\gamma$, Q is a constant from the equation of motion for matter which can be integrated in the form

$$Ea^{p+2} = Q = const. \qquad (11)$$

In Ref.[10] it was shown that addition of a hydrodynamical matter to the scalar field with a potential $V(\varphi) = m^2\varphi^2/2$ can kill the chaos. Here we extend this analysis to less steep potentials. Some of our results are illustrated in Fig.4.

It is interesting that increasing Q acts as increasing φ_0. In Fig.4 three intervals merged at $Q = 0$. When Q increases, the 3-d and 2-nd interval consecutively separate and we return to the chaos typical for $V(\varphi) = m^2\varphi^2/2$. With the further increasing of Q the chaos disappear in a way discussed in [10].

The value of Q corresponding to the chaos disappearing is in general bigger for the less steep potentials with the same effective mass . In Fig.5.(a) this values are plotted for Damour-Mukhanov potentials and the perfect fluid with $\gamma = 0$ ($p = 1$). This particular γ is chosen for a mathematical reason. Namely, it can be seen from (8)-(10) that in this case only the constraint equation is changed in comparison with the initial system (1)-(3). In other words, dynamical equations describing FRW universe with a scalar field and dust matter are formally equivalent to those for scalar field only but with nonzero value of the conserved energy. So, our figure describes not only the physical system under consideration, but also general mathematical properties of (1)-(2).

We recall that for the case $V(\varphi) = m^2\varphi^2/2$ (which is equivalent to the Damour-Mukhanov potential with $q = 2$, the corresponding mass is equal to the effective one $m_{eff} = M_0^2/\varphi_0$) the chaos disappear for $Qm > 0.023m_P$ [10]. To compare with this value, we plotted in Fig.5(a) the values of Qm_{eff} leading to ceasing of the chaos with respect to φ_0 for several q. In units we used the case $q = 2$ corresponds to horizontal line $Qm_{eff} = 1.15$. All other curves have this value as an asymptotic one for large φ_0. With decreasing φ_0 the value Qm_{eff} increases with the rate which is bigger for less steep potentials.

In Fig.5(b) the analogous curve is plotted for asymptotically flat potential (6). The value Qm_{eff} for large φ_0 is the same. For small φ_0 we can estimate Q killing the chaos as the value corresponding to disappearance of the Euclidean region for a flat potential $V(\varphi) = M_0^4$. It can be easily obtained from (4) that
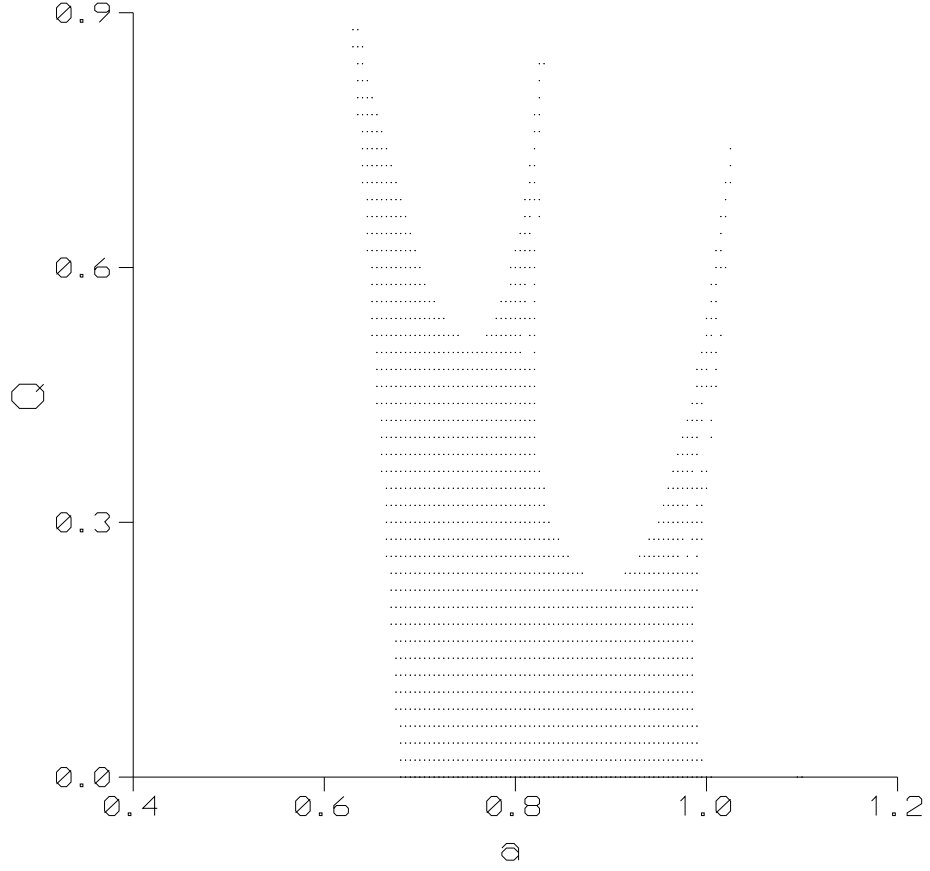
Figure 4: The $\varphi = 0$ cross-section of the bounce intervals for the Damour-Mukhanov potential with $q = 1.0$, $\varphi_0 = 0.1$ depending of the Q. For $Q = 0$ three intervals merges. This plot shows the consecutive separation and further disappearance of the bounce intervals with Q increasing.
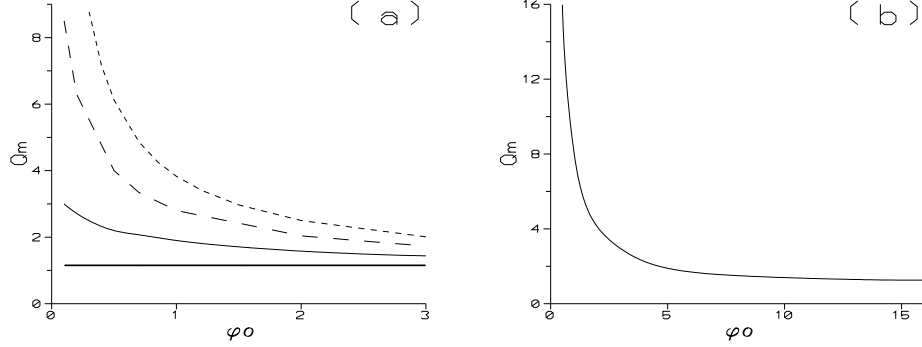
Figure 5: In Fig.5(a) the values of Qm_{eff} killing the chaos for potentials (7) depending on φ_0 are plotted for several q: $q = 2$ (bold line), $q = 1.5$ (solid curve), $q = 1.0$ (long-dashed curve), $q = 0.5$ (short-dashed curve). In Fig.5(b) the values of Qm_{eff} killing the chaos for potentials (6) depending on φ_0 are plotted.

in this case the Euclidean region disappears at

$$Q = \frac{1}{8\sqrt{2\pi^3}} \frac{m_P^3}{M_0^2}$$

and for bigger Q any bounce become impossible. As the potential (6) differs significantly from the flat one only for φ less then φ_0, this approximation appear to be good enough for small φ_0. In the our units it correspond to the curve $Qm_{eff} = 8/\varphi_0$. Intermediate values of φ_0 represent smooth transition between these two asymptotic behaviors.

Acknowledgments

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